APPENDIX H

MINIMIZING THE VSP OBJECTIVE FUNCTIONS

While developing the VSP inversion in Chapter 3, I introduced a global objective function (3.7) and a quadratic objective function (3.10). I now derive the equations that are necessary for gradient methods to minimize these functions.

Let us invert for $g(t)$ over $0 \leq t \leq T$, for $r(x)$ over the recorded depths $\{x_i\}$, and for $\sigma(x)$ over $(0 \leq x \leq X)$ where $X = x_{\max} + (T - x_{\max})/2$. The $x_{\max}$ is the maximum recorded depth. X represents the maximum depth from which a reflection can be made in the recorded time. An assumption that the wavefield is zero at deeper points will not affect the inversion of the earth above; varying the impedance at these depths will not affect the modeled data. We obtain another boundary condition:

$$y = 0 \quad \text{for } x \geq X \quad . \tag{H.1}$$

I use the mathematical techniques of optimal control (Lions [1968]) for the minimization of objective function (3.7). The application to the 1D inversion of seismic waves derives from Bamberger et al. (1982) and Macé and Lailly (1984). These previous applications use the L1 norm instead of the L2 to constrain the impedance in function (3.7).

For each of the iterations I shall linearize the modeled data $y'(x,t)$ with respect to perturbed model parameters.

$$\sigma \frac{\partial^2}{\partial t^2}\delta y - \frac{\partial}{\partial x}\left[\sigma \frac{\partial}{\partial x}\delta y\right] = -\delta\sigma \frac{\partial^2 y}{\partial t^2} + \frac{\partial}{\partial x}\left[\delta\sigma \frac{\partial y}{\partial x}\right] \tag{H.2}$$

$$\sigma \frac{\partial}{\partial x}\delta y = -\delta g - \delta\sigma \frac{\partial y}{\partial x} \quad \text{for } x = 0 \tag{H.3}$$

$$\delta y = \frac{\partial}{\partial t}\delta y = 0 \quad \text{for } t = 0 \tag{H.4}$$

$$\delta y' = (1 + r)\delta y + y \cdot \delta r \tag{H.5}$$

All unperturbed functions remain at reference values.

Let us now calculate the gradient of objective function (3.7).

$$\nabla_\sigma J_1 = \int_0^T \left(\frac{\partial^2 y}{\partial t^2} q + \frac{\partial y}{\partial x}\frac{\partial q}{\partial x}\right) dt + C_\sigma^{-2} \frac{d\,\sigma}{dx} \tag{H.6}$$

$$\nabla_g J_1 = -q \mid_{x=0} + C_g^{-2} g \tag{H.7}$$

$$\nabla_r J_1 = \frac{1}{C_n^{\,2}} \int_0^T y\,(d - y')\,dt + \frac{r}{C_r^{\,2}} \tag{H.8}$$

$q(x,t)$ is determined by

$$\sigma\frac{\partial^2 q}{\partial t^2} - \frac{\partial}{\partial x}[\sigma\frac{\partial q}{\partial x}] = \frac{1}{C_n^2}\sum_{\{x_i\}}(1+r)(d-y')\delta(x-x_i) \tag{H.9}$$

$$q = \frac{\partial q}{\partial t} = 0 \quad \text{for } t = T \tag{H.10}$$

$$\frac{\partial q}{\partial x} = 0 \quad \text{for } x = 0,\ x = X \ . \tag{H.11}$$

The above system is called the adjoint of the linearized system. Condition (H.11) holds because no geophone records at the surface or at depths below X.

Using conventional gradient methods and the gradients (H.6), (H.7), and (H.8) we may iteratively calculate the minima of objective functions (3.7) and (3.10). At any given iteration we have estimates of σ, g, and r and a corresponding estimated wavefield. Subtraction of this wavefield from the data gives an error that the adjoint system will seek to minimize with gradients (H.6), (H.7), and (H.8). For the non-quadratic objective function (3.7), a steepest-descent or Fletcher-Reeves algorithm would perform a line search in order to scale the gradient perturbation. The reference parameters would be perturbed, and a new δy calculated.

Minimization of the quadratic objective function (3.10) also requires the linearized differential system of (H.2), (H.3), (H.4), and (H.5). A steepest-descent or conjugate-gradient minimization requires only a few scalar products for calculation of the perturbation's scale factor. Perturbations calculated from the minimization of objective function (3.10) must always be rescaled somewhat by a line search when (3.7) is minimized. The best scale factor for equation (3.10) generally gives a good upper bound for the value required by (3.7).